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Boundary-layer flow between nodal and saddle points of attachment

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A simplified example of this type of flow was examined in detail by developing two series, eventually matched, one about the nodal point and the other about the saddle point, and also by finite differences, marching from the nodal point to the saddle point. It was found that the results of marching the two series were in agreement with the finite difference method. The series solution near the saddle point is not unique, but numerical evidence indicates that the correct solution is that which has 'exponential decay' at infinity, and that this type of solution, if such exists, automatically emerges when the finite difference method is used.

1. Introduction

Consider a surface and an inviscid irrotational flow as illustrated in figure 1. One would expect the fluid to attach itself to the surface along the line $APQRB$, on which P and R are 'nodal points of attachment' where the fluid on the surface is flowing away from the points, and Q is a 'saddle point of attachment' where in some directions the flow on the surface is towards the point and in others it is away from the point; the word 'attachment' implies that the normal component of velocity is towards the body.

If the origin is taken at P, and x is measured along PQ and y along PC, with z along the normal to the surface, then the local flow near P can be described by

$$U = ax, \quad V = by, \quad W = -(a+b)z,$$

with a and b both positive.

Near Q the flow is of the type (origin now at Q)

$$U = -a^*x^*, \quad V = b^*y^*, \quad W = -(b^*-a^*)z^*,$$

with a^* and b^* positive; we must have $b^* > a^*$, in order that the flow may be of attachment type.

Howarth (1951) studied the flow in the boundary layer in the neighbourhood of P, for a range of values of a/b $(= c)$, and Squire (1957) extended this by means of a series to regions further away from P for two values of c. Work analogous to that of Howarth was done by Davey (1961) in connexion with the point Q

and one of us (Robins 1968) has done the same near to Q as Squire did near to P, for two values of c^* ($= a^*/b^*$, namely 0·5 and 1).

In this study we try to answer the questions: (1) Can the two solutions be joined together, that is, can the series round P be matched to the series around Q—indeed, is it possible to expand in a useful series round Q at all? (2) Is the 'exponential decay' solution of Davey the correct one at Q in these circumstances? By exponential decay we mean that the velocity components at large values of z differ from their limiting values by amounts which are exponentially small.

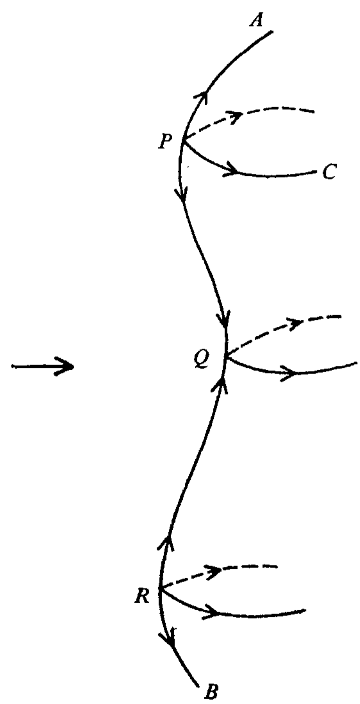

FIGURE 1. Type of flow discussed.

In order to study these questions it was decided to attempt to solve the problem in the case of a greatly simplified flow. First of all the surface is assumed to be a developable surface (or a plane) in which the required type of external irrotational flow is somehow contrived to be of the form

$$U = ax(1-x)+az^2, \quad V = by, \quad W = -(a+b-2ax)z.$$

P is taken to be the point (0, 0, 0) and Q the point (1, 0, 0). Thus near P the flow is of Howarth type and near Q it is of Davey type.

This seems to be the simplest type of external flow that can be devised to exhibit the properties we wish to investigate. It has the advantage that the boundary-layer equations can be transformed into a form which is independent of y, so that the problem can be made quasi-two-dimensional.

We may note that Banks (1967) attacked an analogous problem in which the external flow at the 'start' was different from ours, the velocity components being in our notation

$$U = a(1-x), \quad V = by.$$

Thus the flow near to $x = 1$ was similar to ours. Banks expanded in a series near to $x = 0$ and used a Pohlhausen method to study the behaviour at larger values of x. Tests showed that the Pohlhausen method as applied by Banks was quite accurate and calculations indicated that the velocity profiles were already approaching closely to those of Davey by the time x had reached a value of 0·4.

One implication of our work is indeed that whatever reasonable initial profiles occur at some starting point one will always arrive at a saddle point with the profiles of Davey, provided of course that there is no separation in between. To prove this rigorously would of course be a major task.

It was decided to concentrate on just one ratio a/b $(= c)$ namely 0·25. This was chosen to ensure that no separation occurs between P and Q which will happen if $c > 0{\cdot}42937$ $(= c_0)$. A limited study was also made for a few other values of c including c_0 itself.

Two approaches were made. One was by expanding in a series in x round P and in a series in x^* $(= 1-x)$ round Q. The other was by integrating the equations between P and Q by finite difference methods, marching in the x direction from $x = 0$ to $x = 1$.

2. The equations of motion

The boundary-layer equations for flow over a plane or a developable surface may be written

$$\begin{aligned} u'u'_x + v'u'_y + w'u'_z &= UU_x + VU_y + \nu u'_{zz}, \\ u'v'_x + v'v'_y + w'v'_z &= UV_x + VV_y + \nu v'_{zz}, \\ u'_x + v'_y + w'_z &= 0, \end{aligned}$$

with boundary conditions $u' = v' = w' = 0$ when $z = 0$ and $u' = U, v' = V$ when $z = \infty$.

2.1. *The series expansion*

We write

$$u' = axu(x,\zeta), \quad v' = byv(x,\zeta), \quad w' = (b\nu)^{\frac{1}{2}} w, \quad \zeta = z(b/\nu)^{\frac{1}{2}}, \quad c = a/b,$$

and the equations become

$$\left.\begin{aligned} u_{\zeta\zeta} - wu_\zeta - cxuu_x - cu^2 &= -c(1-x)(1-2x), \\ v_{\zeta\zeta} - wv_\zeta - cxuv_x - v^2 &= -1, \\ w_\zeta + cu + v + cxu_x &= 0, \end{aligned}\right\} \tag{1}$$

with boundary conditions $u = v = w = 0$ for $\zeta = 0$, $u = 1-x$, $v = 1$ for $\zeta = \infty$. The y co-ordinate drops out of the equations, though there is a velocity v' in the y direction. We write, with primes denoting derivatives with respect to ζ,

$$\left.\begin{aligned} u &= f'_0(\zeta) + xf'_1(\zeta) + x^2 f'_2(\zeta) + \dots, \\ v &= g'_0(\zeta) + xg'_1(\zeta) + x^2 g'_2(\zeta) + \dots, \\ w &= -c(f_0 + xf_1 + x^2 f_2 + \dots) - (g_0 + xg_1 + x^2 g_2 + \dots) - cx(f_1 + 2xf_2 - \dots). \end{aligned}\right\} \tag{2}$$

These are substituted into the equations and coefficients of powers of x equated. We obtain a set of pairs of coupled ordinary differential equations all of which, after the first, are linear. The first pair is

$$f_0''' + (g_0 + cf_0) f_0'' + c(1 - f'^2) = 0,$$

$$g_0''' + (g_0 + cf_0) g_0'' + (1 - g_0'^2) = 0,$$

and these are in effect Howarth's equations.

The later equations are

$$f_n''' + (g_0 + cf_0) f_n'' - (n+2) cf_0' f_n' + (n+1) cf_0'' f_n + f_0'' g_n = F_n,$$

$$g_n''' + (g_0 + cf_0) g_n'' - (2g_0' + ncf_0') g_n' + g_0'' g_n + (n+1) cg_0'' f_n = G_n,$$

with F_n and G_n functions of $f_r, f_r', f_r'', g_r, g_r', g_r'' \ldots$ $(r = 0, 1, 2, \ldots, n-1)$.

The boundary conditions are $f_r = g_r = f_r' = g_r' = 0$ for all r when $\zeta = 0$, and $f_0' = g_0' = 1, f_1' = -1$, and the remaining first derivatives zero when $\zeta = \infty$.

For the series in x^* around the saddle point it is easy to show that the equations can be written in a form identically the same as given above but with c replaced by $-c$.

2·2. *The finite difference method*

We write

$$u' = Uu(x, \zeta), \quad v' = Vv(x, \zeta), \quad w' = (b\nu)^{\frac{1}{2}} w, \quad \zeta = (b/\nu)^{\frac{1}{2}} z.$$

The definition of v (but not u) is the same as it was earlier; these slightly differing forms seems to be the most convenient for the particular methods chosen.

The equations become

$$u_{\zeta\zeta} - wu_\zeta - u^2 c(1 - 2x) - cx(1-x) uu_x = -c(1 - 2x),$$

$$v_{\zeta\zeta} - wv_\zeta - v^2 - cx(1-x) uv_x = -1,$$

$$w_\zeta + cx(1-x) u_x + c(1 - 2x) u + v = 0.$$

with boundary conditions $u = v = w = 0$ for $\zeta = 0$, $u = v = 1$ for $\zeta = \infty$. Again the co-ordinate y has dropped out.

At the start we put $x = 0$ and the equations reduce to a form which is essentially that of Howarth. We first solve these by the finite difference method and then 'march' in the x direction taking small steps until we reach $x = 1$.

The method of solution is fully implicit. No assumptions about the values of u_ζ and v_ζ at $\zeta = 0$ (denoted by $(u_\zeta)_0, (v_\zeta)_0$) are made. Here the method differs from the finite difference method of Smith & Clutter (1962) who give them trial values and then adjust these until the outer boundary conditions are satisfied. At each x step we put $u = v = 0$ at $\zeta = 0$ and $u = v = 1$ for a sufficiently large value of ζ and the values of $(u_\zeta)_0$ and $(v_\zeta)_0$ then come directly out of the calculations; these involve essentially the inversion of a tri-diagonal matrix at each step. We do, however, need to assume a pair of velocity profiles at the start of each step; these are usually taken as the values at the previous step except at the very beginning when almost any reasonable assumption will suffice. Once the assumption of profiles at any step has been made the equations for this step are solved by iteration, ceasing when there is no change in the profiles from one iteration to the next.

3. Solution for $c = 0.25$

3.1. *The series*

The first six functions f_0 to f_5, g_0 to g_5 were found both for the nodal point ($c = 0{\cdot}25$) and the saddle point ($c = -0{\cdot}25$), the latter functions being chosen to give exponential decay. The values of $f_r''(0)$ and $g_r''(0)$ are shown in table 1 for each of the values of c. It will be seen that the nodal point series for the skin friction components $(u_\zeta)_0$ and $(v_\zeta)_0$ appear to be well behaved, with the values of $f''(0)$ and $g''(0)$ decreasing with increase in r; the series appear to be converging fairly rapidly when x is less than about 0·8. The saddle point series, however, appear to be asymptotic, reliable results being obtainable only for small values of x^*, say less than 0·3. Despite this it was possible to obtain the skin friction components over the whole range $0 < x < 1$, matching to 3 figure accuracy, or better in the region of $x = 0{\cdot}7$.

The values of the skin friction components are shown in tables 2 and 3, which show how the matching occurred.

	$c = 0{\cdot}25$		$c = -0{\cdot}25$	
n	$f_n''(0)$	$g_n''(0)$	$f_n''(0)$	$g_n''(0)$
0	0·80514	1·24761	0·26795	1·22513
1	−1·2178	−0·0385	+0·4365	−0·0079
2	0·3301	0·0074	−0·8638	0·0473
3	0·0535	0·0044	−0·0159	0·0007
4	0·0178	0·0022	0·4662	−0·0629
5	0·0072	0·0012	−0·0505	0·0116
	$c = 0{\cdot}42937$		$c = -0{\cdot}42937$	
n	$f_n''(0)$	$g_n''(0)$	$f_n''(0)$	$g_n''(0)$
0	0·94678	1·26112	0·00000	1·22732
1	−1·5663	−0·0749	+1·3163	−0·0667
2	0·4508	0·0100	−1·0000	0·0793
3	0·0845	0·0094	−0·8720	0·1631
4	0·0349	0·0068	—	—
5	0·0189	0·0050	—	—
6	0·0118	0·0037	—	—

TABLE 1. Wall derivative of functions f_n and g_n for $c = \pm 0{\cdot}25, \pm 0{\cdot}42937$.

3.2. *The finite difference method*

The initial solution (for $x = 0$) gave Howarth's results and the step-by-step procedure (of length 0·05 in x with intervals in ζ also equal to 0·05) gave no difficulties, and when the point $x = 1$ was reached the solution agreed closely with that of Davey for $c = -0{\cdot}25$. A comparison with the series solution for the skin friction components is shown in tables 2 and 3.

The results make it clear that the ultimate solution at $x = 1$ is indeed Davey's 'exponential decay' solution. Thus both forms of solution and the close agreement between them show that in this case the answers to the two questions

x	Series about $x = 0$		Series about $x = 1$	Finite differences
0·00	1·2476		—	1·2477
0·05	1·2457		—	1·2458
0·10	1·2438		—	1·2439
0·15	1·2420		—	1·2421
0·20	1·2403		—	1·2403
0·25	1·2385		—	1·2386
0·30	1·2369		—	1·2370
0·35	1·2353		—	1·2354
0·40	1·2338		—	1·2338
0·45	1·2323		—	1·2324
0·50	1·2310		—	1·2311
0·55	1·2297		1·2289	1·2298
0·60	1·2285		1·2281	1·2286
0·65	1·2275	Closest	1·2273	1·2276
0·70	1·2266	←→	1·2266	1·2267
0·75	1·2258	match	1·2259	1·2260
0·80	1·2251		1·2253	1·2254
0·85	1·2247		1·2250	1·2250
0·90	—		1·2248	1·2249
0·95	—		1·2249	1·2249
1·0	—		1·2251	1·2252

TABLE 2. Values of $(1/by)/(\nu/b)^{\frac{1}{2}}\,(\partial v'/\partial z)_0$ by series about $x = 0$ and $x = 1$, and by finite differences

x	Series about $x = 0$		Series about $x = 1$	Finite differences
0·00	0·0000		—	0·0000
0·05	0·0373		—	0·0373
0·10	0·0687		—	0·0687
0·15	0·0945		—	0·0945
0·20	0·1150		—	0·1150
0·25	0·1306		—	0·1306
0·30	0·1413		—	0·1413
0·35	0·1477		—	0·1477
0·40	0·1499		—	0·1499
0·45	0·1484		—	0·1484
0·50	0·1434		—	0·1434
0·55	0·1354		0·1378	0·1354
0·60	0·1246	Closest	0·1259	0·1248
0·65	0·1116	←→	0·1123	0·1118
0·70	0·0968	match	0·0973	0·0971
0·75	0·0805		0·0812	0·0811
0·80	0·0634		0·0643	0·0643
0·85	0·0458		0·0471	0·0471
0·90	—		0·0303	0·0303
0·95	—		0·0144	0·0144
1·00	—		0·0000	0·0000

TABLE 3. Values of $(1/a)\,(\nu/b)^{\frac{1}{2}}\,(\partial u'/\partial z)_0$ by series about $x = 0$ and $x = 1$, and by finite differences

proposed earlier are: (1) Starting at the node one ends up with Davey's solution at the saddle point, and a series round this point is possible but is probably an asymptotic series. (2) The correct form of solution at the saddle point is indeed that which has exponential decay.

4. Discussion

The series solution starting at $x = 0$ seems to lead to no difficulties and the series for the components of wall friction appear to be convergent.

Each solution for all orders f_r and g_r is unique and has exponential decay to the appropriate boundary condition at $\zeta = \infty$. Those starting at $x^* = 0$ $(x = 1)$ going backwards are not unique but the solutions chosen were those which exhibited exponential decay. The sums of the series in x^* for the skin friction when $c = -0{\cdot}25$ have a peculiar oscillation in the way they diverge from each other.

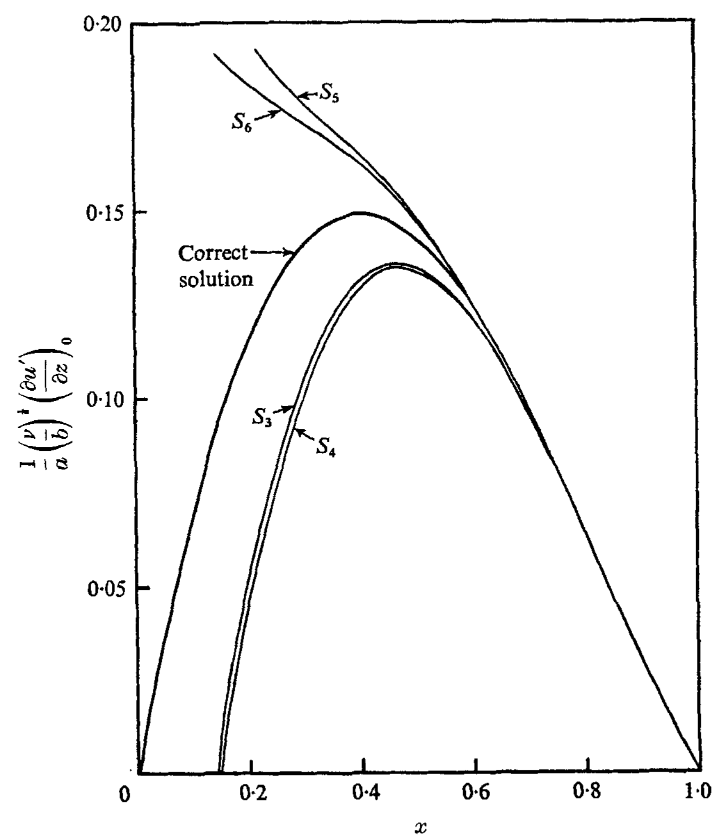

FIGURE 2. Skin friction in x direction. S_r denotes the sum of r terms of the series about $x = 1$.

One usually finds in summing such an expansion as this that the sums for each additional term retained are alternatively one side or other of the 'correct' solution or else approach it smoothly from one side. Here they are in pairs, two in succession going to one side and then two in succession going to the other side. See figures 2 and 3, which also show the 'correct' solution over the whole range.

However, the series solutions about $x = 0$ and $x = 1$ can be matched fairly closely near $x = 0{\cdot}7$, and very closely indeed if the method of accelerating convergence due to Shanks (1955) is applied to the former. This method seems to have no value in the series about $x = 1$. The final results agree closely with the finite difference results.

What appears a little puzzling is that we seem to be able to start at the saddle point end and to go some distance 'upstream'. For comparison a finite difference calculation starting from this end was also tried. The solution duly came out to be the same as Davey's at $x = 1$ (i.e. the start) but on moving upstream only two steps were possible before the method began to fail, as exhibited by the large number of iterations required to give convergence.

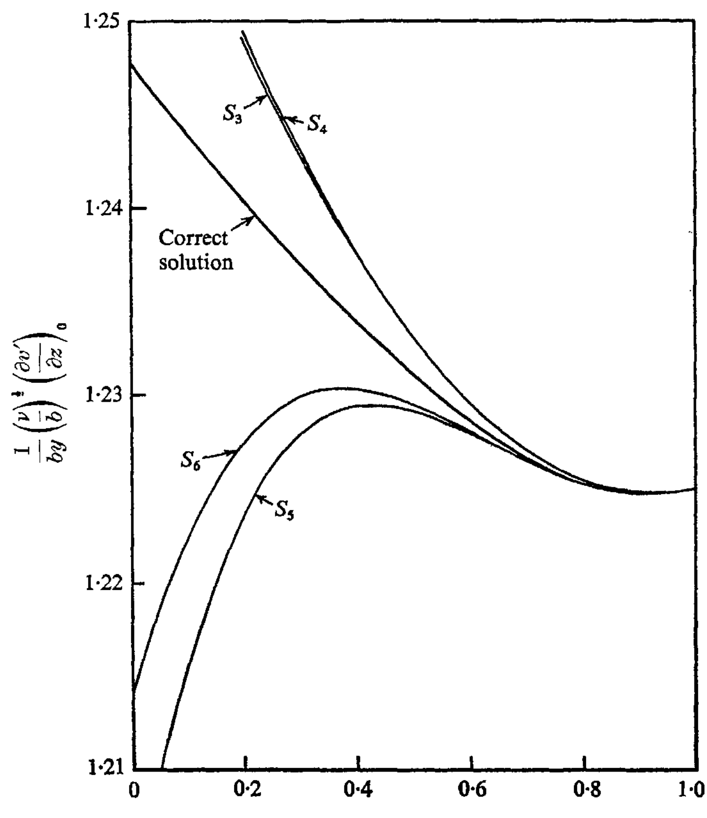

FIGURE 3. Skin friction in y direction. S_r denotes the sum of r terms of the series about $x = 1$.

5. Eigensolutions

As we have shown, it seems possible to start at the saddle point and to travel a certain distance upstream without appreciable error. This suggests that the eigensolutions in the expansion around $x^* = 0$ have not yet become important within this distance.

To investigate the eigensolution near to $x^* = 0$ we write in (1)

$$u = u_0 + u_1 x + \ldots + x^\lambda(U_0 + U_1 x + \ldots) + \ldots,$$
$$v = v_0 + v_1 x + \ldots + x^\lambda(V_0 + V_1 x + \ldots) + \ldots,$$
$$w = w_0 + w_1 x + \ldots + x^\lambda(W_0 + W_1 x + \ldots) + \ldots,$$

where for convenience we have replaced x^* by x; the series

$$u_0 + u_1 x + \ldots, \quad v_0 + v_1 x + \ldots, \quad w_0 + w_1 x + \ldots$$

are the same as those in (2).

Substitute in (1) and equate the coefficients of x^λ and we find

$$\left.\begin{aligned} U_{0\zeta\zeta} - w_0 U_{0\zeta} - W_0 u_{0\zeta} - c(2+\lambda)\, U_0 u_0 &= 0, \\ V_{0\zeta\zeta} - w_0 V_{0\zeta} - W_0 v_{0\zeta} - (u_0 c\lambda + 2v_0)\, V_0 &= 0, \\ W_{0\zeta} + c(1+\lambda)\, U_0 + V_0 &= 0, \end{aligned}\right\} \tag{3}$$

with boundary conditions $U_0 = V_0 = W_0 = 0$ for $\zeta = 0$, $U_0 = V_0 = 0$ for $\zeta = \infty$.

These homogeneous linear equations with homogeneous boundary conditions have solutions for any positive value of λ but there is only a discrete set of values of λ if U_0 and V_0 are to have exponential decay at infinity, cf. Libby & Fox (1963). We note that U_0 and V_0 may have constant multipliers, one of which is arbitrary.

Equations (3) were solved for $c = -0{\cdot}25$. To find the eigenvalues a trial and error method was used to estimate the value of λ which gave exponential decay. By this means it was found that the first eigenvalue was $\lambda_1 = 9{\cdot}810$. Higher values were difficult to find without a more refined procedure than that undertaken, and the next value was only found roughly. It was $\lambda_2 = 15{\cdot}9$. The equations are coupled through W_0 and the computations are long and tedious. The first eigenvalue for $c = -0{\cdot}42937$ was also computed and the value is $\lambda_1 = 4{\cdot}81$.

The arbitrary factor of (say) U_0 (and the corresponding one for V_0) in the solution lead to the expected indeterminancy in the solution upstream of the saddle point $x^* = 0$, but the power of x^* for the lowest eigenvalue (9·810) might have led one to believe that one could have got further upstream than $x^* = 0{\cdot}3$ in the original series without sensible error. There is however no knowing how great the arbitrary multiplier of U_0 should be.

It was indeed hoped to find an approximate value for this multiplier by the present work, but such a high value of λ_1, together with the asymptotic nature of the original series, seems to preclude this.

The work in fact shows two kinds of indeterminancy. The first is due to the arbitrary multiplier of the eigensolutions, and this is genuine and to be expected. The second is due to the asymptotic nature of the series, and presumably arises from the kind of mathematics used in the solution.

6. Exponential decay at infinity

Brown (1966) considers a flow which at large distances x downstream approximates to Hartree (1937) flow with $\beta = -0{\cdot}15$, and gives some numerical evidence for believing that at $x = \infty$ the correct limiting solution is the one with exponential decay. In the problem considered here the limiting position is at $x = 1$ and we have shown likewise that the limiting solution at $x = 1$ is the one of Davey's which has exponential decay, and this is the solution which emerges from the finite difference method. It may be conjectured that the finite difference solution will *always* pick out the exponential decay solution if such exists. To test this in a special case we solved Hartree's equation

$$f''' + ff'' + \beta(1 - f'^2) = 0.$$

with $\beta = -0{\cdot}15$ by this method.

As already described the procedure in the finite difference method is to *assume* a velocity profile and iterate on the equations of motion, thus changing the profile on each iteration until there is no more change. It might be thought possible, therefore, that when the solution is not unique, the final solution would depend on the initial profile chosen. Our numerical experiments suggest that this is not so, at least for a wide range of initial profiles, and we always seem to obtain the Hartree solution. In this case ($\beta = -0{\cdot}15$) there is another solution which has exponential decay, namely the so-called lower branch solution in which there is reversed flow near to the wall (Stewartson 1954). In an attempt to obtain this solution we assumed an initial profile which had reversed flow near to the wall, but we still obtained the Hartree solution. To test further we started with an assumed profile very near indeed to the known lower-branch solution for this value of β (Rosenhead 1963, p. 250). This time the reversed flow and the boundary-layer thickness grew indefinitely, so that the method failed, but at least it did show that the lower-branch solution is in some sense unstable, and this leads one to believe that the solution cannot exist in a real flow. One might argue that the iterative procedure in the finite difference method is in some sense a marching procedure (perhaps, though, marching in time) and this makes it seem likely that the finite difference solution may give the correct stable solution and that other solutions, although satisfying the differential equation, cannot exist in a real steady flow (see appendix A).

As we have pointed out the finite difference method always seems to give the exponential decay solution if one exists. If two exist it gives only one of them. The next question is then: if an exponential decay solution does not exist, but only one with algebraic decay, will the method give this solution? Algebraic decay solutions do occur as Brown & Stewartson (1965) pointed out; one example is the solution of the equation

$$f''' - ff'' + 4(1 - f'^2) = 0,$$

which is given in Rosenhead (1963), the decay being like ζ^{-8}. This was also tested (with a straight line initial profile) and the solution of Rosenhead did indeed emerge.

7. Other values of c

7.1. *The series*

It was thought worth while to investigate further Davey's critical case, that is $c = c_0 = -0{\cdot}4294$ for which $f_0''(0) = 0$, since it was thought that special features might emerge for this critical value. It was found that an improved value for

c_0 was $-0{\cdot}42937$, one more decimal place than was given by Davey. No real difference did emerge from this investigation. The divergence of the saddle point series is more marked and only the first four terms of the series were calculated, because the values of $f_n''(0)$ and $g_n''(0)$ began to increase rapidly for $n > 3$; it appears again that the saddle point series are asymptotic, good results being obtainable for small values of x^* from the first four terms of the series. Seven terms of the nodal point series were calculated for this value of c, and after applying Shank's (1955) transformation to the terms of the nodal point series it was again found possible to match the series for the skin friction components, this time in the region of $x = 0{\cdot}85$.

The values of $f_r''(0)$ and $g_r''(0)$ for this case are given in table 1.

7.2. *Finite differences*

Calculations were tried for $c = 0{\cdot}75$ for which Davey's solution gives reversed flow. Here the method proceeded satisfactorily as far as $x = 0{\cdot}75$ but then began to break down. It was found that $(u_\zeta)_0$ was decreasing rapidly and if $\{(u_\zeta)_0\}^{1{\cdot}3}$ was plotted against x a very nearly straight line emerged. Thus the value of x at which the x component of the skin friction vanishes can be found by extending this line to the point where $(u_\zeta)_0 = 0$. If we call this value x_s we find that $x_s = 0{\cdot}773$ and so this is the point where separation of the flow is to be expected. Thus we find that $(u_\zeta)_0$ behaves like $(x_s - x)^{0{\cdot}77}$ with $x_s = 0{\cdot}773$ near to separation. However, without further refinement, which was not undertaken, it is not possible to affirm that the index is 0·77 to any great degree of accuracy. The values of $(v_\zeta)_0$ showed no sign of a singularity and certainly were not behaving as though $(v_\zeta)_0$ would vanish at the separation point.

Of course if separation takes place in this manner the assumed external flow will no longer be valid and the computation is no longer applicable to the physical situation. Davey suggests that the saddle point now becomes a nodal point and that the separated flow is convected downstream round the body as a vortex sheet which will probably roll up.

A similar computation was made for $c = c_0 = 0{\cdot}42937$ and it was found that one arrived at $x = 1$ without difficulty and that the solution there was close to that of Davey, which has $(u_\zeta)_0 = 0$. The skin friction curves again fitted closely to those found from the matched series solutions. This time there was no sign of any singularity as $x = 1$ was approached, and $(u_\zeta)_0$ approached zero quite smoothly and almost linearly.

We also tried a value of c nearer to c_0 than 0·75, namely $c = 0{\cdot}5$, which should just give separation. This time the zero of $(u_\zeta)_0$ was at $x_s = 0{\cdot}926$ and the singularity was less severe than with $c = 0{\cdot}75$; $(u_\zeta)_0$ behaves like $(x_s - x)^{0{\cdot}86}$ approximately. Once more there was no singular behaviour by $(v_\zeta)_0$.

8. Conclusions

A simplified flow between a nodal point and a saddle point is considered, by means of series expansions about the two points, and also by a marching finite difference procedure. It is found that the two series can be patched together reasonably well, though that about the saddle point seems to be asymptotic. The first two eigenvalue solutions of the latter are investigated.

It is found that, provided separation does not take place, the particular solution of Davey which has exponential decay appears to be the correct one at the saddle point. (The solutions are not unique in this region.) Examples are given to suggest that the finite difference procedure employed gives a unique solution which turns out to be the exponential decay solution if one exists.

Two cases in which the flow separates are examined. At separation the skin friction in the x direction is singular and vanishes but there is no sign of vanishing or singular skin friction in the y direction.

Appendix A. The effect of linearization

In our iterative finite difference scheme we guess an initial profile and improve it step-by-step after linearizing the equation of motion.

Suppose, for instance, that we are solving by iteration the equation

$$-uu_x + u_{yy} = 0 \tag{A 1}$$

and at one step the value of u is $u^{(0)}$ and at the next step it is $u^{(1)}$. One common way of linearizing the equation would be to write it

$$-u^{(0)}u_x^{(1)} + u_{yy}^{(1)} = 0 \tag{A 2}$$

where $u^{(0)}$ is known and $u^{(1)}$ is now to be found. Equation (A 2) is now linear and is solved by finite differences again and again using an improved $u^{(0)}$ at each step to obtain an improved $u^{(1)}$.

Now we may write

$$-u^{(0)}u_x^{(1)} + u_{yy}^{(1)} = -u^{(1)}u_x^{(1)} + u_{yy}^{(1)} + u_x^{(1)}(u^{(1)} - u^{(0)})$$

and so solving (A 2) is equivalent to solving (A 1) with an additional term; in fact to solving

$$-uu_x + u_{yy} + u_x \frac{\partial u}{\partial t} = 0$$

interpreting each iteration step as a time step with unit time between steps. This is the sense in which we say our procedure could imply a marching in time.

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